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J. Symbolic Computation (2001) **32**, 231–253
doi:10.1006/jsco.2001.0472
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Symbolic Reachability Computation for Families of Linear Vector Fields

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The control paradigm of physical processes being supervised by digital programs has lead to the development of a theory of hybrid systems combining finite state automata with differential equations. One of the most important problems in the verification of hybrid systems is the reachability problem. Even though the computation of reachable spaces for finite state machines is well developed, computing the reachable space of a differential equation is difficult. In this paper, we present the first known families of linear differential equations with a decidable reachability problem. This is achieved by posing the reachability computation as a quantifier elimination problem in the decidable theory of the reals. We illustrate the applicability of our approach by performing computations using the packages REDLOG and QEPCAD. Such symbolic computations can be incorporated in computer-aided verification tools for purely discrete systems, resulting in verification tools for hybrid systems with linear differential equations.

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1. Introduction

In recent years there has been growing interest in the analysis and design of embedded systems which merge physical processes with information systems. A typical example is the control of physical systems, such as cars, aircraft, and robots, by digital programs. As a consequence, modeling formalisms for embedded systems must contain both continuous and discrete models.

Hybrid systems (Maler, 1997; Henzinger and Sastry, 1998; Vaandrager and van Schuppen, 1999) combine discrete event systems with differential equations in a manner that is ideal for the modeling, analysis, and design of embedded systems. The safety criticality of many applications requires the use of formal methods to guarantee that an unsafe region of the state space is not reachable from a set of initial conditions. This makes the *reachability problem* for hybrid systems very important.

Computing reachable sets for purely discrete systems has matured to the level of developing a variety of computer-aided verification tools that perform either *model checking*

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or *automated theorem proving*. On the other hand, computing reachable sets for purely continuous systems is difficult. Due to the infinite cardinality of continuous state spaces, the crucial issue for computing reachable spaces of differential equations is *decidability*. It is therefore immediate that the computability of reachable sets for differential equations is vital to the development of formal verification tools for hybrid systems. In particular, the decidability results of Alur and Dill (1994) and Henzinger *et al.* (1995b) have led to the development of model checking tools for hybrid automata (Henzinger *et al.*, 1994; Alur *et al.*, 1995), such as KRONOS (Daws *et al.*, 1996) and HYTECH (Henzinger *et al.*, 1995a). KRONOS can compute reachable sets for timed automata, that is, automata extended with differential equations of the form $\dot{\xi} = 1$, whereas HYTECH can handle differential inclusions of the form $A\dot{\xi} \leq b$. On the other hand, STEP (Bjorner *et al.*, 1996) is a deductive verification tool for hybrid systems for the same class of differential inclusions.

However, many hybrid systems usually require much more complicated continuous models than those that can be handled by the above mentioned tools. An important class of differential equations is linear systems of the form $\dot{\xi} = A\xi + Bu$. Recently, in Lafferriere *et al.* (1999a,b), we have shown that the reachability problem for classes of hybrid systems with linear differential equations of the form $\dot{\xi} = A\xi + Bu$ is decidable. These results are based on two key concepts from model theory, namely *o-minimality* (van den Dries, 1998) and *quantifier elimination* (Tarski, 1951; Collins, 1975; Arnon *et al.*, 1984; Collins and Hong, 1991; Weispfenning, 1993).

Whereas o-minimality enables us in Lafferriere *et al.* (2000) to determine classes of hybrid systems with a decidable reachability problem, quantifier elimination is our engine for computing reachable sets of linear differential equations. In this paper, we focus on the computation of reachable sets for *families* of linear control systems of the form $\dot{\xi} = A\xi + Bu$, where u belongs to a set $\mathcal{U}$ of possible inputs, and therefore generalize our previous results. Our approach consists of characterizing the set of reachable states as a predicate in the theory of the ordered field of real numbers. A quantifier-free characterization of the reachable sets can then be obtained by quantifier elimination tools such as REDLOG (Dolzman and Sturm, 1997) and QEPCAD (Collins and Hong, 1991). Clearly, such symbolic computations can be embedded in existing verification tools resulting in meaningful tools for hybrid systems with linear differential equations. Furthermore, these results are significant in their own right given the wide applicability of linear systems in control theory.

The use of quantifier elimination in control theory goes as far back as Anderson *et al.* (1975), where it was used to obtain an algorithmic solution to the problem of stabilization by static output feedback. More recently, a number of researchers have used quantifier elimination in testing stability of linear systems (Hong *et al.*, 1997), robust feedback control (Dorato *et al.*, 1997), trajectory tracking of non-linear control systems (Jirstrand, 1997), and analysis of discrete-time polynomial systems (Nešić, 1998; Anai and Kaneko, 1999). However, the problem of computing the *exact* reachable set of linear vector fields had not been addressed.

Methods for exact computation of reachable sets should be contrasted with those based on computing *approximations*. These methods over- or under-approximate reachable sets using a variety of set representations such as polyhedra, level sets, or ellipsoids. Approximate reachability computations rely on numerical methods for Hamilton–Jacobi equations (Mitchell and Tomlin, 2000), ellipsoidal calculus (Kurzhanski and Varaiya, 2000), flow-pipe approximations (Chutinam and Krogh, 1999), and polygonal computations (Dang and Maler, 1998). As a result, approximate methods are, in principle, applicable to larger classes of continuous systems, such as general linear and non-linear systems, while sacrificing precision. Future research in this area will clearly integrate symbolic and numeric methods.

The outline of this paper is as follows: in Section 2 we review the relevant notions from mathematical logic and model theory that will be used throughout the paper. In Section 3 we use these notions to determine three distinct classes of families of linear control systems whose reachable set can be computed using quantifier elimination. Each class is accompanied by examples that illustrate such reachability computations. Finally, Section 4 contains conclusions and issues for further research.

2. Preliminaries

In this section we give a brief review of mathematical logic and model theory. The interested reader is referred to van Dalen (1994) and Chang and Keisler (1990) for a detailed exposition to the subject .

2.1. languages and formulae

A *language* is a set of symbols separated in three groups: relations, functions and constants. The sets $\mathcal{P} = \{<, +, -, 0, 1\}$, $\mathcal{R} = \{<, +, -, \cdot, 0, 1\}$, and $\mathcal{R}_{\exp} = \{<, +, -, \cdot, e^x, 0, 1\}$ are examples of languages where $<$ (less than) is the relation, $+$ (plus), $-$ (minus), $\cdot$ (product) and e^x (exponentiation) are the functions, and 0 (zero) and 1 (one) are the constants.

Let $\mathcal{V} = \{x, y, z, x_0, x_1, \ldots\}$ be a countable set of *variables*. The set of *terms* of a language is inductively defined as follows. A term θ is a variable, a constant, or $F(\theta_1, \ldots, \theta_m)$, where F is a m-ary function and θ_i, $i = 1, \ldots, m$ are terms. For instance, $x - 2y + 3$ and $x + yz^2 - 1$ are terms of $\mathcal{P}$ and $\mathcal{R}$, respectively. In other words, terms of $\mathcal{P}$ are linear expressions and terms of $\mathcal{R}$ are polynomials with integer coefficients. Notice that integers are the only numbers allowed in expressions (for example the integer 2 is an abbreviation for $1 + 1$). Rational coefficients can also be allowed since terms involving rational coefficients can be rescaled to terms involving only integer coefficients.

The *atomic formulae* of a language are of the form $\theta_1 = \theta_2$, or $R(\theta_1, \ldots, \theta_n)$, where θ_i, $i = 1, \ldots, n$ are terms and R is an n-ary relation. For example, $xy > 0$ and $x^2 + 1 = 0$ are atomic formulae of $\mathcal{R}$. The set of *(first-order) formulae* is recursively defined as follows: Every atomic formula ϕ is a (first-order) formula, and if ϕ_1 and ϕ_2 are formulae and x is a variable, then $\phi_1 \wedge \phi_2$, $\neg\phi_1$, $\forall x : \phi_1$ or $\exists x : \phi_1$ are formulae. Symbols $\forall$ (for all) and $\exists$ (there exists) are the quantifiers. Examples of $\mathcal{R}$-formulae are:

$$\forall x\, \forall y : xy > 0 \tag{2.1}$$

$$\exists x : x^2 - 2 = 0 \tag{2.2}$$

$$\exists w : xw^2 + yw + z = 0 \wedge x \neq 0. \tag{2.3}$$

The occurrence of a variable in a formula is *free* if it is not inside the scope of a quantifier; otherwise, it is *bound*. For example, x, y, and z are free and w is bound in (2.3). We often write $\phi(x_1, \ldots, x_n)$ to indicate that $x_1, \ldots, x_n$ are the free variables of the formula ϕ. A *sentence* of $\mathcal{R}$ is a formula with no free variables. Formulae (2.1) and (2.2) are sentences.

2.2. MODELS AND THEORIES

A *model* of a language consists of a non-empty set S and an interpretation of the relations, functions and constants. For example, $(\mathbb{R}, <, +, -, \cdot, 0, 1)$ and $(\mathbb{Q}, <, +, -, \cdot, 0, 1)$, are *models* of $\mathcal{R}$ with the usual meaning of the symbols. For example, the symbol $+$ stands for function of addition.

Every sentence of a language will be either true or false in a given model. For instance, formula (2.2) is true in the model $(\mathbb{R}, <, +, -, \cdot, 0, 1)$ but false in $(\mathbb{Q}, <, +, -, \cdot, 0, 1)$. Formulae that are not sentences may hold for some assignments of values to the free variables but not for others. For instance, formula (2.3) holds in $(\mathbb{R}, <, +, -, \cdot, 0, 1)$ for the assignment $(1, 1, 0)$ of (x, y, z) but not for $(1, 0, 1)$.

We say that a set $Y \subseteq S^n$ is *definable* in a language if there exists a formula $\phi(x_1, \ldots, x_n)$ such that:

$$Y = \{(a_1, \ldots, a_n) \in S^n \mid \phi(a_1, \ldots, a_n)\}.$$

For example, over $(\mathbb{R}, <, +, -, \cdot, 0, 1)$, the formula $x^2 - 2 = 0$ defines the set $\{\sqrt{2}, -\sqrt{2}\}$.

Two formulae $\phi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are *equivalent* in a model, denoted by $\phi \equiv \psi$, if for every assignment $(a_1, \ldots, a_n)$ of $(x_1, \ldots, x_n)$, $\phi(a_1, \ldots, a_n)$ is true if and only if $\psi(a_1, \ldots, a_n)$ is true. Equivalent formulae define the same set.

A *theory* is a subset of sentences. Any model of a language defines a theory: *the set of all sentences which are true in the model*. For the sake of simplicity, we denote by $\mathcal{R}(\mathbb{R})$ the theory obtained by interpreting the language $\mathcal{R}$ over the model $(\mathbb{R}, <, +, -, \cdot, 0, 1)$. In other words, $\mathcal{R}(\mathbb{R})$ is the set of all true assertions about the set of real numbers when viewed as an *ordered field*.

2.3. DECIDABILITY AND QUANTIFIER ELIMINATION

Given a theory, it is important to determine whether a sentence of the language belongs to the theory. Tarski (1951) showed that $\mathcal{R}(\mathbb{R})$ is *decidable*, and therefore there is a computational procedure that, given any $\mathcal{R}$-sentence ϕ, decides whether ϕ belongs to $\mathcal{R}(\mathbb{R})$.

The decision procedure is based on the elimination of the quantifiers. Over the reals, every formula $\phi(x_1, \ldots, x_n)$ of $\mathcal{R}$ is equivalent to a formula $\psi(x_1, \ldots, x_n)$ without quantifiers. Moreover, there is an algorithm that transforms ϕ into ψ by *eliminating the quantifiers*. For example, formula (2.3) is equivalent to:

$$4xz - y^2 \leq 0 \wedge x \neq 0. \tag{2.4}$$

Notice that the assignment $(1, 1, 0)$ of (x, y, z) satisfies (2.4) whereas $(1, 0, 1)$ does not.

Quantifier elimination implies that every $\mathcal{R}$-definable set $Y \subseteq \mathbb{R}^n$ is definable without quantifiers. Moreover, the decidability of $\mathcal{R}(\mathbb{R})$ implies that the algorithm for eliminating the quantifiers also provides a computational procedure that terminates in a finite number of steps for checking whether a definable set Y is empty: $Y = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid \phi(y_1, \ldots, y_n)\} = \emptyset$ if and only if the sentence $\exists y_1 \ldots \exists y_n : \phi(y_1, \ldots, y_n)$ is equivalent to the quantifier-free formula $false$. Tarski's original algorithm has been dramatically improved over the years, which has allowed the development of computational tools that perform quantifier elimination, namely QEPCAD (Arnon *et al.*, 1984; Collins and Hong, 1991) and REDLOG (Weispfenning, 1993).

Tarski asked if the decidability result for $\mathcal{R}(\mathbb{R})$ could be extended to the theory of reals with exponentiation [i.e. the set of sentences that hold in $(\mathbb{R}, <, +, -, \cdot, e^x, 0, 1)$].

This theory is known to be decidable provided Schanuel's conjecture holds (Macintyre and Wilkie, 1996). Also, van den Dries (1984) proved that there are formulae of this theory that are not equivalent to a quantifier-free formula.

Nevertheless, in Section 3 we identify extensions of the language $\mathcal{R}$ involving the exponential function, and even the functions $\sin(\cdot)$ and $\cos(\cdot)$, where quantifiers can be eliminated, and the resulting quantifier-free formula is in $\mathcal{R}$, thus yielding a decision procedure. The search for such classes is motivated by our main goal, namely to apply symbolic (computer-algebra based) techniques for analyzing hybrid systems with linear vector fields.

3. Reachability Computations for Linear Vector Fields

A large class of continuous processes is modeled by linear control systems which are differential equations of the following form

$$\dot{\xi} = A\xi + Bu \tag{3.1}$$

where $\xi(t) \in \mathbb{R}^n$ is the state of the system at time t, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are the system matrices, and $u : \mathbb{R} \longrightarrow \mathbb{R}^m$ is a piecewise continuous function which is called the control input. Given an initial state $x = \xi(0)$ at time zero, and a control input u, the solution of the above differential equation for any time $t \geq 0$ is

$$\xi(t) = \Phi(x, u, t) = e^{At}x + \int_0^t e^{A(t-\tau)}Bu(\tau)\, d\tau \tag{3.2}$$

where the matrix exponential e^{At} is defined by the series,

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k. \tag{3.3}$$

For all subsequent analysis, it suffices to consider linear control systems of the form

$$\dot{\xi} = A\xi + u \tag{3.4}$$

with $u : \mathbb{R} \longrightarrow \mathbb{R}^n$, since given a system of the form (3.1) we can simply perform the substitution $u' = Bu$, which will result in a system of the form (3.4). Furthermore, we will assume that the control input u belongs to a set $\mathcal{U}$ of input functions, where

$$\mathcal{U} = \left\{ u = [u_1, \ldots, u_n]^T \;\mid\; \bigwedge_{j=1}^{n} u_j(t) = \sum_{l=1}^{r} b_{jl} p_l(t) \;\; \wedge \bigwedge_{l=1}^{r} \Gamma(b_{jl}) \right\} \tag{3.5}$$

where $\Gamma(b_{jl})$ is an $\mathcal{R}$-formula, and $p_l(t)$ are some basis functions (to be determined later). Therefore, $\mathcal{U}$ consists of linear combinations of these basis functions, where the coefficients of the linear combination satisfy some semi-algebraic constraint.

A *family* of linear vector fields is defined as a tuple $\mathcal{F} = (A, \mathcal{U})$. Given a family $\mathcal{F}$ we say that a state y is reachable from a state x if there exists a control input $u \in \mathcal{U}$ and a $t \geq 0$ such that $y = \Phi(x, u, t)$. Our goal in this paper is to solve the following reachability problem.

PROBLEM 3.1. (REACHABILITY COMPUTATION) Given a family $\mathcal{F} = (A, \mathcal{U})$ of linear vector fields, compute all states that are reachable from an $\mathcal{R}$-definable set of initial states.

Given a family $\mathcal{F}$ and sets $Y, Z \subseteq \mathbb{R}^n$, we denote by $\mathrm{Pre}_{\mathcal{F}}(Y \mid Z)$ the set of all $x \in \mathbb{R}^n$ that can reach some $y \in Y$ by some trajectory of $\mathcal{F}$ without ever leaving the set Z. More precisely,

$$\begin{aligned}\mathrm{Pre}_{\mathcal{F}}(Y \mid Z) = \{x \in \mathbb{R}^n \mid \exists y\ \exists u\ \exists t :\ & y \in Y \wedge u \in \mathcal{U} \wedge t \geq 0 \\ & \wedge \Phi(x,u,t) = y \wedge \forall s : 0 \leq s \leq t \Rightarrow \Phi(x,u,s) \in Z\}. \end{aligned} \tag{3.6}$$

The set $\mathrm{Pre}_{\mathcal{F}}(Y \mid Z)$ is the *backward reachability set* of Y given by $\mathcal{F}$, subject to Z. Dually, we define the *forward reachability set* of Y given by $\mathcal{F}$, subject to Z, denoted $\mathrm{Post}_{\mathcal{F}}(Y \mid Z)$, as follows,

$$\begin{aligned}\mathrm{Post}_{\mathcal{F}}(Y \mid Z) = \{x \in \mathbb{R}^n \mid \exists y\ \exists u\ \exists t :\ & y \in Y \wedge u \in \mathcal{U} \wedge t \geq 0 \\ & \wedge \Phi(y,u,t) = x \wedge \forall s : 0 \leq s \leq t \Rightarrow \Phi(y,u,s) \in Z\}. \end{aligned} \tag{3.7}$$

Problem 3.1 will be solved for several families of linear control vector fields for which $\mathrm{Pre}_{\mathcal{F}}(Y \mid Z)$ and $\mathrm{Post}_{\mathcal{F}}(Y \mid Z)$ are definable in $\mathcal{R}$. Quantifier elimination can then be used in order to obtain quantifier free formulae for $\mathrm{Pre}_{\mathcal{F}}(Y \mid Z)$ and $\mathrm{Post}_{\mathcal{F}}(Y \mid Z)$.

In order to simplify the notation of the following presentation, we assume that the set Z is equal to $\mathbb{R}^n$. From the subsequent discussion, it will be clear how to include the more general cases in a straightforward manner. Furthermore, we omit the subscript $\mathcal{F}$ whenever the family of vector fields is clear from the context, and we also use the following notation:

$$\Psi(u,t) = \int_0^t e^{A(t-\tau)} u(\tau)\, d\tau. \tag{3.8}$$

Under these assumptions, and without loss of generality, in the remainder of this section we will consider the simpler formulae

$$\mathrm{Pre}(Y) = \{x \in \mathbb{R}^n \mid \exists y\ \exists u\ \exists t :\ y \in Y \wedge u \in \mathcal{U} \wedge t \geq 0 \wedge \Phi(x,u,t) = y\} \tag{3.9}$$

$$\mathrm{Post}(Y) = \{x \in \mathbb{R}^n \mid \exists y\ \exists u\ \exists t :\ y \in Y \wedge u \in \mathcal{U} \wedge t \geq 0 \wedge \Phi(y,u,t) = x\}. \tag{3.10}$$

Depending on the eigenstructure of the matrix A we identify three different families of linear vector fields for which formulae (3.9) and (3.10) are either *definable by* or *equivalent to* formulae in the decidable theory of the reals. For ease of presentation, we will develop the analysis and the proofs for $\mathrm{Pre}(Y)$. Nevertheless, all the results also apply to $\mathrm{Post}(Y)$.

3.1. NILPOTENT MATRICES

Let $A \in \mathbb{Q}^{n \times n}$ be a *nilpotent* matrix, that is $A^n = 0$. Then the series for the matrix exponential is simply

$$e^{At} = \sum_{k=0}^{n-1} \frac{t^k}{k!} A^k.$$

Therefore, in this case, the expression $e^{At}x$ is clearly a vector of polynomials in $\mathbb{Q}[x,t]$, where each component can be written as follows:

$$(e^{At}x)_i = \sum_{j=1}^{n} (e^{At})_{ij} x_j = \sum_{j=1}^{n} \left(\sum_{k=0}^{n-1} \frac{t^k}{k!} A^k \right)_{ij} x_j = \sum_{k=0}^{n-1} \sum_{j=1}^{n} (A^k)_{ij} x_j \frac{t^k}{k!} = \sum_{k=0}^{n-1} \gamma_{ik}(x) t^k$$

where $\gamma_{ik}(x)$ is an $\mathcal{R}$-term.

The class of control inputs allowed depends on the eigenstructure. For a nilpotent matrix A, let $\mathcal{U}$ be defined as in equation (3.5) where the basis functions are of the form $p_l(t) = t^l$. Therefore $\mathcal{U}$ consists of polynomials in t, whose coefficients satisfy semi-algebraic constraints. Let $u = [u_1, \ldots, u_n]^T$ be a particular input in $\mathcal{U}$, that is, for all $1 \le j \le n$:

$$u_j(t) = \sum_{l=0}^{r} b_{jl} t^l \tag{3.11}$$

where the vector of coefficients $b = (b_{10}, \ldots, b_{nr})$ satisfies $\Gamma(b)$.

For a particular $u \in \mathcal{U}$, it follows from equation (3.8) that after integration, $\Psi(u,t)$ is a vector of polynomials in $\mathbb{Q}[b,t]$. More precisely, the ith component of $\Psi(u,t)$ is

$$\begin{aligned}
\Psi(u,t)_i &= \int_0^t (e^{A(t-\tau)}u(\tau))_i \; d\tau = \int_0^t \left(\sum_{k=0}^{n-1} \frac{(t-\tau)^k}{k!} A^k u(\tau) \right)_i \; d\tau \\
&= \sum_{k=0}^{n-1} \int_0^t \frac{(t-\tau)^k}{k!} (A^k u(\tau))_i \; d\tau = \sum_{k=0}^{n-1} \int_0^t \frac{(t-\tau)^k}{k!} \sum_{j=1}^{n} (A^k)_{ij} u_j(\tau) \; d\tau \\
&= \sum_{k=0}^{n-1} \int_0^t \frac{(t-\tau)^k}{k!} \sum_{j=1}^{n} \sum_{l=0}^{r} (A^k)_{ij} b_{jl} \tau^l \; d\tau \\
&= \sum_{k=0}^{n-1} \int_0^t p_{ik}(b,t,\tau) \; d\tau, \quad \text{with } p_{ik} \in \mathbb{Q}[b,t,\tau] \\
&= \sum_{k=0}^{s} \psi_{ik}(b) t^k
\end{aligned}$$

for some $s \in \mathbb{N}$ and $\mathcal{R}$-terms $\psi_{ik}(b)$, for all $0 \le k \le s$. Hence, $\Phi(x,u,t)$ can be written component-wise as follows, for all $1 \le i \le n$:

$$\Phi(x,u,t)_i = \sum_{k=0}^{n-1} \gamma_{ik}(x) t^k + \sum_{k=0}^{s} \psi_{ik}(b) t^k = \sum_{k=0}^{q} \phi_{ik}(b,x) t^k$$

where $q = \max(n-1, s)$ and for all $1 \le k \le q$, $\phi_{ik}(b,x)$ is an $\mathcal{R}$-term. This proves the following lemma:

Lemma 3.2. *If $\mathcal{F}$ is a family of vector fields such that*

(1) $A \in \mathbb{Q}^{n \times n}$ is a nilpotent matrix, and
(2) $\mathcal{U}$ is defined by equation (3.5) *with $p_l(t) = t^l$,*

then $\Phi(x,u,t)$ is a vector of polynomials in $\mathbb{Q}[x,b,t]$, and thus definable in $\mathcal{R}$.

Let Y be an $\mathcal{R}$-definable set, that is, there exists an $\mathcal{R}$-formula Φ_Y such that $Y = \{y \in \mathbb{R}^n \mid \Phi_Y(y)\}$. Then Lemma (3.2) implies that expression (3.9) for $\mathrm{Pre}(Y)$ can be re-written as follows:

$$\mathrm{Pre}(Y) = \left\{ x \in \mathbb{R}^n \mid \exists y \; \exists b \; \exists t : \; \Phi_Y(y) \wedge \Gamma(b) \wedge t \ge 0 \wedge \bigwedge_{i=1}^{n} y_i = \sum_{k=0}^{q} \phi_{ik}(b,x) t^k \right\}. \tag{3.12}$$

This proves the following proposition:

Proposition 3.3. *Let $Y \subseteq \mathbb{R}^n$ be an $\mathcal{R}$-definable set and $\mathcal{F} = (A,\mathcal{U})$ be a family of vector fields such that*

(1) $A \in \mathbb{Q}^{n\times n}$ is a nilpotent matrix, and
(2) $\mathcal{U}$ is defined by equation (3.5) with $p_l(t) = t^l$.

Then, $\mathrm{Pre}(Y)$ *and* $\mathrm{Post}(Y)$ *are definable in $\mathcal{R}$. Moreover, they are computable.*

Example 3.1. Consider the control linear vector field given by the nilpotent matrix $A \in \mathbb{Q}^{3\times 3}$ and $\mathcal{U} = \{u\}$ defined as follows:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} 8t - t^2 \\ 2 - 3t \\ -1 \end{bmatrix},$$

and consider the set of initial states:

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 3 \le x_2 \le 5 \wedge x_3 = 5\}.$$

It can easily be checked that:

$$\mathrm{Post}(X) = \Big\{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid \exists x_1\ \exists x_2\ \exists x_3\ \exists t : t \ge 0 \wedge 3 \le x_2 \le 5 \wedge x_3 = 5 \wedge \\ y_1 = x_1 - x_2\ t + (x_3 + 6)\ \frac{t^2}{2} \wedge y_2 = x_2 + (2 - x_3)\ t - t^2 \wedge y_3 = x_3 - t\Big\}.$$

Using Redlog to perform quantifier elimination we get that:

$$\mathrm{Post}(X) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid -37 \le y_2 + y_3^2 - 13y_3 \le -35 \wedge y_3 \le 5\}.$$

The computation time for this example (as well as for all other examples in this paper) is negligible. The two-dimensional projection on the variables y_2 and y_3 of the set $\mathrm{Post}(X)$ is depicted in Figure 1 for $2 \le y_3 \le 5$.

Example 3.2. Let $A \in \mathbb{Q}^{3\times 3}$ be as in Example 3.1 and $\mathcal{U}$ be such that $u \in \mathcal{U}$ iff:

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} bt - t^2 \\ a - 3t \\ -1 \end{bmatrix}, \qquad \text{with } a^2 + b^2 = 1.$$

The set of points in $\mathbb{R}^3$ that can reach the final set $Y = \{(0,0,0)\}$ following the flows defined by the linear control vector field $(A,\mathcal{U})$ is:

$$\mathrm{Pre}(Y) = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \exists a\ \exists b\ \exists t : t \ge 0 \wedge a^2 + b^2 = 1 \wedge \\ 0 = x_1 - x_2 t + (x_3 + b - a)\frac{t^2}{2} \wedge 0 = x_2 + (a - x_3)t - t^2 \wedge 0 = x_3 - t\Big\}.$$

Using Redlog to eliminate the quantifiers we obtain:

$$\mathrm{Pre}(Y) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = x_3 = 0 \vee \\ x_3 > 0 \wedge 4x_1^2 - 4x_1x_2x_3 - 4x_1x_3^3 + 2x_2^2x_3^2 - 2x_2x_3^4 + 5x_3^6 - x_3^4 = 0\}$$

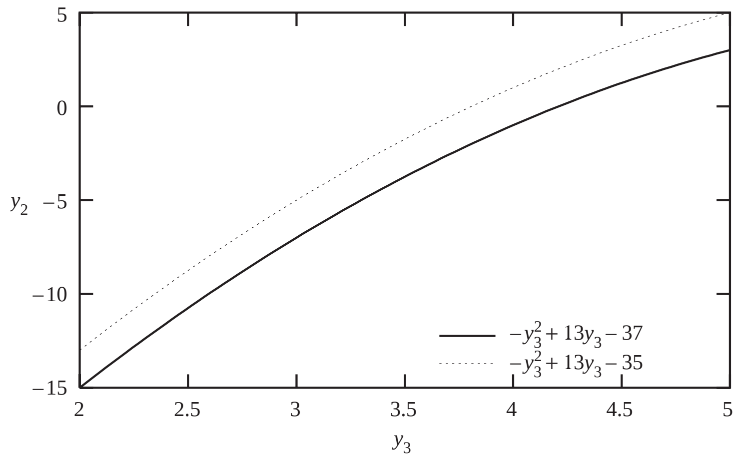

Figure 1. Reachability computation for Example 3.1.

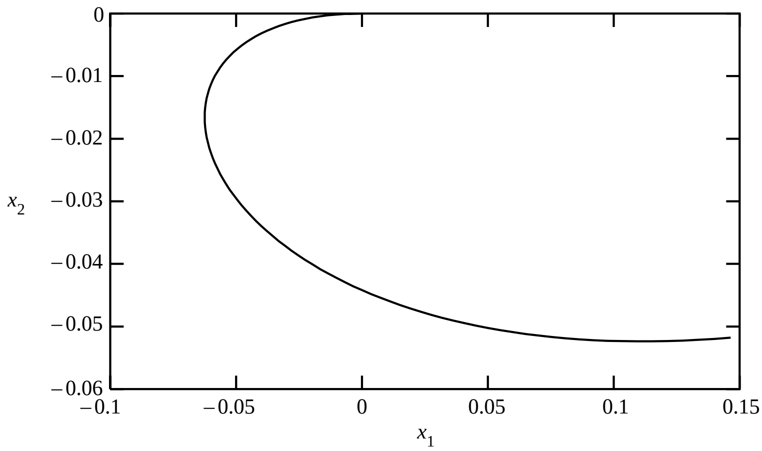

Figure 2. Reachability computation for Example 3.2.

with the following instantiation of parameters for $x_3 > 0$:

$$a = \frac{-x_2 + 2x_3^2}{x_3} \qquad b = \frac{-2x_1 + x_2 x_3 + x_3^3}{x_3^2}.$$

For instance, for $a = b = \frac{\sqrt{2}}{2}$ we obtain:

$$x_1 = -\frac{\sqrt{2}}{2}x_3^2 + \frac{3}{2}x_3^3 \qquad x_2 = -\frac{\sqrt{2}}{2}x_3 + 2x_3^2.$$

The set $\mathrm{Pre}(Y)$ is depicted in Figure 2 for these values with $0 \leq x_3 \leq \frac{1}{2}$.

Example 3.3. A mobile vehicle is located at an initial position (x_0, y_0) with initial velocity $(v_{x_0}, v_{y_0}) = (0, 0)$ and initial acceleration $(a_{x_0}, a_{y_0}) = (0, 0)$ and it is desired that it reaches the target position (x_F, y_F) with velocity $(v_{x_F}, v_{y_F}) = (0,0)$ and final acceleration $(a_{x_F}, a_{y_F}) \in [a_{\min}, a_{\max}]^2$ in a given time T. The motion of the vehicle is modeled as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{v_x} \\ \dot{v_y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_x(t) \\ a_y(t) \end{bmatrix}$$

where $a_x(t), a_y(t) \in \mathcal{U}$ are such that:

$$a_x(t) = a_2 t^2 + a_1 t + a_0$$
$$b_x(t) = b_2 t^2 + b_1 t + b_0.$$

Let the initial position be $p_0 = (x_0, y_0, v_{x_0}, v_{y_0}) = (1,1,0,0)$ and the final position be $p_F = (x_F, y_F, v_{x_F}, v_{y_F}) = (\frac{3}{4}, \frac{15}{4}, 0, 0)$. We ask whether it is possible to drive the vehicle from p_0 to p_F in time $T = 50$ starting with an initial acceleration $a_x(0) = a_0 = a_y(0) = b_0 = 0$ and ending with a final acceleration $a_x(50) = a_y(50) \in [-\frac{1}{10}, \frac{1}{10}]$, while following a continuous trajectory where the x-velocity is positive and the y-velocity is negative (i.e. $x(t)$ is monotone increasing and $y(t)$ is monotone decreasing in the interval $[0,50]$). It is straightforward to formalize this problem in our framework and we omit it here. We use the quantifier elimination algorithm implemented in Redlog. The result obtained is the following expression on the parameters a_2, a_1, b_2, and b_1:

$$\begin{aligned} & 3a_1 + 100a_2 \geq 0 \wedge 3b_1 + 100b_2 \leq 0 \\ \wedge\ & (b_1 b_2 \geq 0 \vee b_2 = 0 \vee 3b_1 b_2 + 100 b_2^2 \leq 0 \wedge 3b_1 + 100b_2 < 0 \vee a_1 b_2^2 - a_2 b_1 b_2 \geq 0 \\ & \quad \wedge b_2 \geq 0 \wedge (a_1 b_2 - a_2 b_1 \neq 0 \vee a_2 \leq 0)) \\ \wedge\ & (a_1 a_2 \geq 0 \vee a_2 = 0 \vee 3a_1 a_2 + 100 a_2^2 \leq 0 \wedge 3a_1 + 100a_2 > 0 \vee a_1 a_2 b_2 - a_2^2 b_1 \geq 0 \\ & \quad \wedge a_2 \leq 0 \wedge (a_1 b_2 - a_2 b_1 \neq 0 \vee b_2 \geq 0)). \end{aligned}$$

We existentially quantify the parameters and use the quantifier elimination algorithm with the Redlog answer to obtain the following instances of the parameters that satisfy the constraint:

$$a_1 = \frac{33}{62500} \qquad a_2 = -\frac{99}{6250000} \qquad b_1 = -\frac{3}{62500} \qquad b_2 = \frac{9}{6250000}.$$

The trajectory obtained is depicted in Figure 3.

3.2. Diagonalizable matrices with rational eigenvalues

A matrix A is said to be *diagonalizable* if a diagonal matrix D and an invertible matrix T exist such that $A = TDT^{-1}$. The matrix D is formed with the eigenvalues of A along the diagonal. The columns of T form a basis of eigenvectors of A. If all the eigenvalues of A are rational numbers, then the matrices D, T and T^{-1} belong to $\mathbb{Q}^{n \times n}$. In this case, we have that

$$(e^{At})_{ij} = (e^{TDT^{-1}t})_{ij} = \left(T \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} T^{-1} \right)_{ij} = \sum_{k=1}^{n} a_{ijk} e^{\lambda_k t} \qquad (3.13)$$

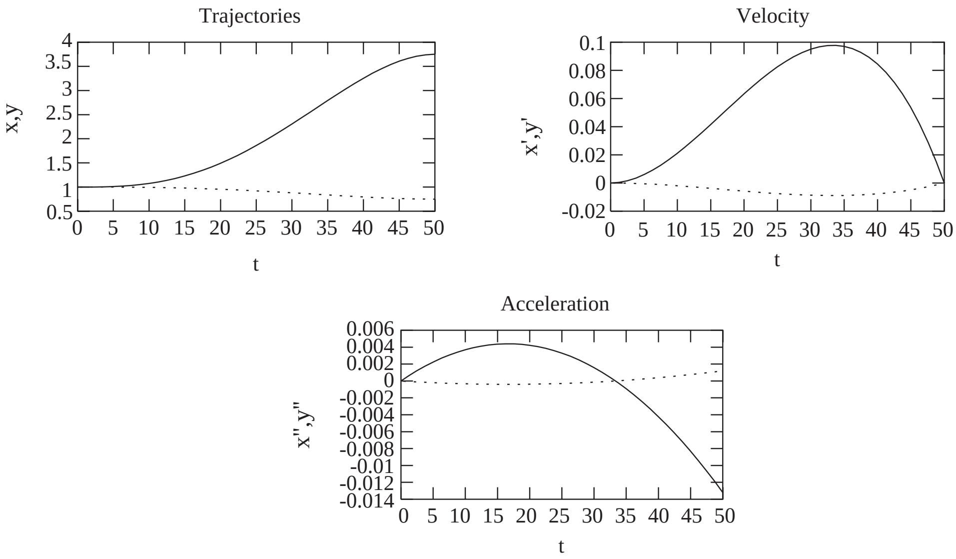

Figure 3. Computed trajectory for Example 3.3.

with $a_{ijk} \in \mathbb{Q}$, and $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ is the set of eigenvalues of A. Then, the vector $e^{At}x$ can be written component-wise as follows:

$$(e^{At}x)_i = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{ijk} e^{\lambda_k t} \right) x_j = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{ijk} x_j \right) e^{\lambda_k t} = \sum_{k=1}^{n} \gamma_{ik}(x) e^{\lambda_k t}$$

for all $1 \leq i \leq n$, where for all $1 \leq k \leq n$, $\gamma_{ik}(x)$ is an $\mathcal{R}$-term.

The class of control inputs $\mathcal{U}$ allowed in this case is given by equation (3.5) where the basis functions are now of the form $p_l(t) = e^{\mu_l t}$ with $\mu_l \notin \Lambda$, $\mu_l \in \mathbb{Q}$. Therefore $\mathcal{U}$ consists of linear combinations of exponentials whose coefficients satisfy semi-algebraic constraints. Furthermore, a *resonance condition* is imposed, namely, that if $\mu_l \in \mathbb{Q}$ is an eigenvalue of A, then $e^{\mu_l t}$ cannot belong to $\mathcal{U}$. The need for such a condition will become clear later. Consider now $u \in \mathcal{U}$, which for all $1 \leq j \leq n$ is of the form

$$u_j(t) = \sum_{h=1}^{r} b_{jh} e^{\mu_{jh} t} \tag{3.14}$$

where the vector of coefficients $b = (b_{11}, \ldots, b_{nr})$ satisfies $\Gamma(b)$, and for all $1 \leq h \leq r$, $\mu_{jh} \notin \Lambda$, $\mu_{jh} \in \mathbb{Q}$.

From equation (3.13) we have that each entry of the matrix e^{At} is a linear combination of the functions $e^{\lambda_i t}$ with λ_i an eigenvalue of A. We now prove that the entries of the matrix $e^{-A\tau}u(\tau)$ are linear combinations of the functions $e^{(\mu_{jh} - \lambda_i)t}$. It follows that $\Psi(u,t)$ is such that, for all $1 \leq i \leq n$,

$$\begin{aligned}
\Psi(u,t)_i &= \int_0^t (e^{A(t-\tau)}u(\tau))_i \; d\tau = \int_0^t (e^{At}e^{-A\tau}u(\tau))_i \; d\tau \\
&= \sum_{k=1}^{n} (e^{At})_{ik} \int_0^t (e^{-A\tau}u(\tau))_k \; d\tau = \sum_{k=1}^{n} (e^{At})_{ik} \int_0^t \sum_{j=1}^{n} (e^{-A\tau})_{kj} u_j(\tau) \; d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} (e^{At})_{ik} \int_0^t \sum_{j=1}^{n} \left(\sum_{l=1}^{n} a_{kjl} e^{-\lambda_l \tau} \sum_{h=1}^{r} b_{jh} e^{\mu_{jh}\tau} \right) \, d\tau \\
&= \sum_{k=0}^{n} (e^{At})_{ik} \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{h=1}^{r} a_{kjl} b_{jh} \int_0^t e^{(\mu_{jh}-\lambda_l)\tau} \, d\tau, \quad \text{with } (\mu_{jh}-\lambda_l) \neq 0 \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{h=1}^{r} (e^{At})_{ik} \frac{a_{kjl} b_{jh}}{\mu_{jh}-\lambda_l} (e^{(\mu_{jh}-\lambda_l)t} - 1) \\
&= \sum_{k=1}^{s} \psi_{ik}(b) e^{\nu_k t}
\end{aligned}
$$

for some $s \in \mathbb{N}$, and for all $1 \leq k \leq s$, $\nu_k \in \mathbb{Q}$, and $\psi_{ik}(b)$ is an $\mathcal{R}$-term.

Thus, $\Phi(x,u,t)$ can be written component-wise as follows, for all $1 \leq i \leq n$:

$$
\Phi(x,u,t)_i = \sum_{k=1}^{n} \gamma_{ik}(x) e^{\lambda_k t} + \sum_{k=1}^{s} \psi_{ik}(b) e^{\nu_k t} = \sum_{k=1}^{q} \phi_{ik}(b,x) e^{\eta_k t} = \widehat{\Phi}(x,b,e^t)_i
$$

for some $q \in \mathbb{N}$, where for all $1 \leq k \leq q$, $\eta_k \in \mathbb{Q}$, and $\phi_{ik}(b,x)$ are $\mathcal{R}$-terms. Now, for each $1 \leq k \leq q$, let d_k denote the denominator of η_k and let $d = \prod_{k=1}^{q} d_k$. We assume that the η_k are in reduced form, with positive denominators. Then, $d > 0$ and for each $1 \leq k \leq q$, we write $\rho_k = \eta_k d$. By using the change of variable $z = e^{\frac{t}{d}}$ we have that:

$$
\widetilde{\Phi}(x,b,z)_i = \widehat{\Phi}(x,b,e^t)_i[z/e^{\frac{t}{d}}] = \sum_{k=1}^{q} \phi_{ik}(b,x) z^{\rho_k}, \qquad \text{with } z \geq 1.
$$

Now, let $I_+ = \{k \mid \rho_k > 0\}$, $I_- = \{k \mid \rho_k < 0\}$, and $I_0 = \{k \mid \rho_k = 0\}$. Then,

$$
\begin{aligned}
\widetilde{\Phi}(x,b,z)_i &= \overline{\Phi}(x,b,z,w)_i \\
&= \sum_{k \in I_+} \phi_{ik}(b,x)\, z^{\rho_k} + \sum_{k \in I_-} \phi_{ik}(b,x)\, w^{-\rho_k} + \sum_{k \in I_0} \phi_{ik}(b,x), \quad \text{with } zw = 1.
\end{aligned}
$$

This proves the following lemma:

Lemma 3.4. *If $\mathcal{F}$ is a family of vector fields such that*

(1) $A \in \mathbb{Q}^{n \times n}$ is diagonalizable with real, rational eigenvalues,
(2) $\mathcal{U}$ is defined by equation (3.5) with $p_l(t) = e^{\mu_l t}$, $\mu_l \notin \Lambda$, $\mu_l \in \mathbb{Q}$

then there exists a vector $\overline{\Phi}$ of polynomials $\overline{\Phi}_i \in \mathbb{Q}[x,b,z,w]$ such that for all $t \geq 0$ and $u \in \mathcal{U}$, $\Phi(x,u,t) \equiv \overline{\Phi}(x,b,z,w)$ with $z \geq 1$ and $zw = 1$.

Notice that, if we do not impose the resonance conditions and some constant μ_{jh} in (3.14) is allowed to be an eigenvalue of A, after integration, some term of Ψ is of the form $\psi(b)t$. It follows that, in this case, Φ cannot be expressed as a polynomial $\overline{\Phi}$ since the change of variable to z will introduce logarithms.

Let $Y = \{y \in \mathbb{R}^n \mid \Phi_Y(y)\}$. Then Lemma (3.4) implies that $\mathrm{Pre}(Y)$ is equivalent to:

$$
\begin{aligned}
\mathrm{Pre}(Y) = \Big\{ & x \in \mathbb{R}^n \mid \exists y\, \exists b\, \exists z\, \exists w : \Phi_Y(y) \wedge \Gamma(b) \\
& \wedge z \geq 1 \wedge zw = 1 \wedge \bigwedge_{i}^{n} y_i = \overline{\Phi}(x,b,z,w)_i \Big\}. \qquad (3.15)
\end{aligned}
$$

The previous discussion proves the following proposition.

PROPOSITION 3.5. *Let $Y \subseteq \mathbb{R}^n$ be an $\mathcal{R}$-definable set and $\mathcal{F} = (A, \mathcal{U})$ be a family of vector fields such that*

(1) $A \in \mathbb{Q}^{n\times n}$ is diagonalizable with real, rational eigenvalues, and
(2) $\mathcal{U}$ is defined by equation (3.5) with $p_l(t) = e^{\mu_l t}$, $\mu_l \notin \Lambda$, $\mu_l \in \mathbb{Q}$.

Then, Pre(Y) and Post(Y) are definable in $\mathcal{R}$. Moreover, they are computable.

EXAMPLE 3.4. Consider the linear control vector field given by the diagonal matrix $A \in \mathbb{Q}^{2\times 2}$ and $\mathcal{U} = \{u\}$ defined as follows:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}.$$

The corresponding flow is:

$$\Phi(x_1, x_2, u, t) = \begin{bmatrix} x_1 e^{2t} \\ (x_2 - \frac{1}{2})e^{-t} + \frac{1}{2}e^t \end{bmatrix}.$$

Let X and Y be defined as follows:

$$X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 3\} \qquad Y = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 4 \wedge y_2 = 3\}.$$

Then, $X \cap \mathrm{Pre}(Y)$ is:

$$\begin{aligned} X \cap \mathrm{Pre}(Y) = \{(x_1, x_2) \in \mathbb{R}^2 \mid \exists y_1\ \exists y_2\ \exists t : t \geq 0 \wedge x_2 \geq 3 \\ \wedge y_1 = 4 \wedge y_2 = 3 \wedge x_1 e^{2t} = y_1 \wedge (2x_2 - 1)e^{-t} + e^t = 2y_2\}. \end{aligned}$$

After substitution and simplification we obtain:

$$\begin{aligned} X \cap \mathrm{Pre}(Y) = \{(x_1, x_2) \in \mathbb{R}^2 \mid \exists y_1\ \exists z : z \geq 1 \wedge x_2 \geq 3 \\ \wedge y_1 = 4 \wedge x_1 z^2 = y_1 \wedge (2x_2 - 1) + z^2 = 6z\}. \end{aligned}$$

Using QEPCAD to eliminate the quantifiers we obtain:

$$\begin{aligned} X \cap \mathrm{Pre}(Y) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 3 \\ \wedge 4x_1^2 x_2^2 - 4x_1^2 x_2 + 16x_1 x_2 + x_1^2 - 152x_1 + 16 = 0\}. \end{aligned}$$

The set $X \cap \mathrm{Pre}(Y)$ is depicted in Figure 4.

EXAMPLE 3.5. Consider the linear control vector field given by the diagonal matrix $A \in \mathbb{Q}^{2\times 2}$ of Example 3.4 and let $\mathcal{U}$ be such that for all $u \in \mathcal{U}$:

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} -ae^{\frac{1}{2}t} \\ ae^t \end{bmatrix}, \qquad \text{with } a \geq 0.$$

The corresponding flow is:

$$\Phi(x_1, x_2, u, t) = \begin{bmatrix} x_1 e^{2t} + \frac{2}{3}a\big(-e^{2t} + e^{\frac{1}{2}t}\big) \\ x_2 e^{-t} + \frac{1}{2}a\big(e^t - e^{-t}\big) \end{bmatrix}.$$

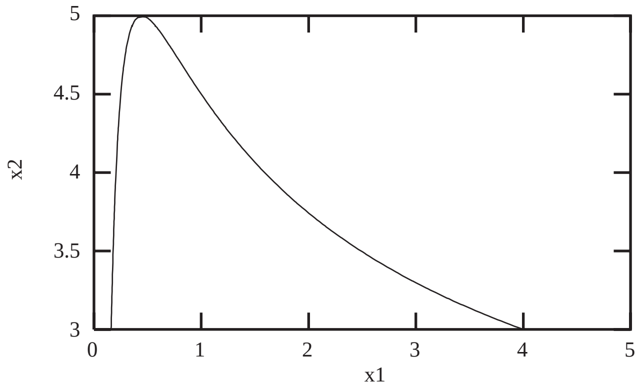

Figure 4. Reachability computation for Example 3.4.

Let the initial set be $X = \{(0,0)\}$. Then, Post(X) is:

$$\mathrm{Post}(X) = \Big\{(y_1, y_2) \in \mathbb{R}^2 \mid \exists a\ \exists t : 0 \leq a \leq 1 \wedge t \geq 0$$
$$\wedge y_1 = \frac{2}{3}a\big(-e^{2t} + e^{\frac{1}{2}t}\big) \wedge y_2 = \frac{1}{2}a(e^t - e^{-t})\Big\}.$$

After substitution and simplification we obtain:

$$\mathrm{Post}(X) = \Big\{(y_1, y_2) \in \mathbb{R}^2 \mid \exists a\ \exists z : 0 \leq a \leq 1 \wedge z \geq 1$$
$$\wedge y_1 = \frac{2}{3}a(-z^4 + z) \wedge y_2 z^2 = \frac{1}{2}a(z^4 - 1)\Big\}.$$

We were not able to eliminate the quantifiers using Redlog or Qepcad alone. To overcome the problem, we first use Redlog to eliminate a and we obtain:

$$\mathrm{Post}(X) = \{(y_1, y_2) \in \mathbb{R}^2 \mid \exists z : z \geq 1 \wedge 3y_1(z^3 + z^2 + z + 1) + 4y_2(z^5 + z^4 + z^3) = 0\}.$$

Using Qepcad to eliminate z we obtain:

$$\mathrm{Post}(X) = \{(y_1, y_2) \in \mathbb{R}^2 \mid (y_2 > 0 \wedge y_1 + y_2 \leq 0) \vee (y_2 < 0 \wedge y_1 + y_2 \geq 0) \vee 4y_2 + 3y_1 = 0\}.$$

3.3. Purely imaginary eigenvalues

Let A be a diagonalizable matrix such that all its eigenvalues are purely imaginary, with rational imaginary part. More precisely, $\Lambda = \{\pm \mathbf{i}\lambda_1, \ldots, \pm \mathbf{i}\lambda_m\}$ with $n = 2m$ and $\lambda_k \in \mathbb{Q}$ for all $1 \leq k \leq m$. Then A is similar to a matrix in a special block-diagonal form, the real Jordan form, that is, an invertible matrix T and a block diagonal matrix $D \in \mathbb{Q}^{n \times n}$ exist, such that $A = TDT^{-1}$, where:

$$D = \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_m \end{bmatrix}$$

where the blocks $D_1, \ldots, D_m$ are of the form:

$$D_k = \begin{bmatrix} 0 & -\lambda_k \\ \lambda_k & 0 \end{bmatrix}.$$

We know that:

$$e^{D_k t} = \begin{bmatrix} \cos(\lambda_k t) & -\sin(\lambda_k t) \\ \sin(\lambda_k t) & \cos(\lambda_k t) \end{bmatrix}.$$

Then, the matrix e^{At} is such that, for all $1 \leq i, j \leq n$,

$$(e^{At})_{ij} = \left(T \begin{bmatrix} e^{D_1 t} & & \\ & \ddots & \\ & & e^{D_m t} \end{bmatrix} T^{-1} \right)_{ij} = \sum_{k=1}^{m} a_{ijk} \cos(\lambda_k t) + c_{ijk} \sin(\lambda_k t) \tag{3.16}$$

with $a_{ijk}, c_{ijk}, \lambda_k \in \mathbb{Q}$, for all $1 \leq k \leq m$. Then, the vector $e^{At}x$ can be written component-wise as follows:

$$\begin{aligned}
(e^{At}x)_i &= \sum_{j=1}^{n} \left(\sum_{k=1}^{m} a_{ijk} \cos(\lambda_k t) + c_{ijk} \sin(\lambda_k t) \right) x_j \\
&= \sum_{k=1}^{m} \left(\sum_{j=1}^{n} a_{ijk} x_j \right) \cos(\lambda_k t) + \sum_{k=1}^{m} \left(\sum_{j=1}^{n} c_{ijk} x_j \right) \sin(\lambda_k t) \\
&= \sum_{k=1}^{m} \gamma_{ik}^{a}(x) \cos(\lambda_k t) + \gamma_{ik}^{c}(x) \sin(\lambda_k t).
\end{aligned}$$

Given this eigenstructure of the matrix A, the class of control inputs $\mathcal{U}$ allowed in this case is given by equation (3.5) where the basis functions are now of the form $p_l(t) = \sin(\mu_l t)$ and $p_l(t) = \cos(\mu_l t)$ with $\mathbf{i}\mu_l \notin \Lambda$, $\mu_l \in \mathbb{Q}$. Therefore $\mathcal{U}$ consists of linear combinations of sinusoids whose coefficients satisfy semi-algebraic constraints. Furthermore, as in Section 3.2, and for similar reasons, a resonance condition is imposed, namely, that if $\mathbf{i}\mu_{jh}$ is an eigenvalue of A, then the corresponding sinusoids cannot be in $\mathcal{U}$. Consider now $u \in \mathcal{U}$, which for all $1 \leq j \leq n$ is of the form

$$u_j(t) = \sum_{h=1}^{r} \alpha_{jh} \cos(\mu_{jh} t) + \beta_{jh} \sin(\mu_{jh} t) \tag{3.17}$$

where the vector of coefficients $b = (\alpha_{11}, \ldots, \beta_{nr})$ satisfies $\Gamma(b)$, and for all $1 \leq h \leq r$, $\mathbf{i}\mu_{jh} \notin \Lambda$, $\mu_{jh} \in \mathbb{Q}$.

Because of the form of the input $u \in \mathcal{U}$, the entries of $e^{-A\tau}u(\tau)$ are linear combinations of products of sines and cosines. More precisely, for all $1 \leq k \leq n$:

$$\begin{aligned}
&\left(e^{-A\tau}u(\tau)\right)_k \\
&= \sum_{j=1}^{n} (e^{-A\tau})_{kj} u_j(\tau) \\
&= \sum_{j=1}^{n} \left(\sum_{l=1}^{m} a_{ijl} \cos(\lambda_l \tau) + c_{ijl} \sin(\lambda_l \tau) \right) \left(\sum_{h=1}^{r} \alpha_{jh} \cos(\mu_{jh} \tau) + \beta_{jh} \sin(\mu_{jh} \tau) \right)
\end{aligned}$$

$$= \sum_{j=1}^{n}\sum_{l=1}^{m}\sum_{h=1}^{r} a_{ijl}\alpha_{jh}\cos(\lambda_l\tau)\cos(\mu_{jh}\tau) + a_{ijl}\beta_{jh}\cos(\lambda_l\tau)\sin(\mu_{jh}\tau)$$
$$+c_{ijl}\alpha_{jh}\sin(\lambda_l\tau)\cos(\mu_{jh}\tau) + c_{ijl}\beta_{jh}\sin(\lambda_l\tau)\sin(\mu_{jh}\tau).$$

Using standard product formulae these can be re-written as linear combinations of $\sin((\mu_{jh} \pm \lambda_l)\tau)$ and $\cos((\mu_{jh} \pm \lambda_l)\tau)$:

$$\begin{aligned}(e^{-A\tau}u(\tau))_k &= \sum_{j=1}^{n}\sum_{l=1}^{m}\sum_{h=1}^{r} d_{ijl}\cos((\mu_{jh}+\lambda_l)\tau) + e_{ijl}\cos((\mu_{jh}-\lambda_l)\tau)\\ &\qquad + f_{ijl}\sin((\mu_{jh}+\lambda_l)\tau) + g_{ijl}\sin((\mu_{jh}-\lambda_l)\tau)\\ &= \sum_{j=1}^{s} p_{kj}\cos(\eta_{kj}\tau) + q_{kj}\sin(\eta_{kj}\tau)\end{aligned}$$

for some $s \in \mathbb{N}$, and $0 \neq \eta_{kj} \in \mathbb{Q}$ and $p_{kj}, q_{kj} \in \mathbb{Q}[b]$, for all $1 \leq j \leq s$ and $1 \leq k \leq n$. Now, $\Psi(u,t)$ can be written component-wise as follows, for all $1 \leq i \leq n$:

$$\begin{aligned}&\Psi(u,t)_i\\ &= \int_0^t \left(e^{A(t-\tau)}u(\tau)\right)_i \, d\tau = \int_0^t \left(e^{At}e^{-A\tau}u(\tau)\right)_i \, d\tau = \sum_{k=1}^{n}\left(e^{At}\right)_{ik}\int_0^t \left(e^{-A\tau}u(\tau)\right)_k \, d\tau\\ &= \sum_{k=1}^{n}\left(e^{At}\right)_{ik}\int_0^t \sum_{j=1}^{s} p_{kj}\cos(\eta_{kj}\tau) + q_{kj}\sin(\eta_{kj}\tau)\, d\tau\\ &= \sum_{k=1}^{n}\left(e^{At}\right)_{ik}\sum_{j=1}^{s} v_{kj}\cos(\eta_{kj}t) + w_{kj}\sin(\eta_{kj}t) + z_{kj}, \qquad \text{with } v_{kj}, w_{kj}, z_{kj} \in \mathbb{Q}[b]\\ &= \sum_{l=1}^{q} \psi_{il}^a(b)\cos(\nu_{il}t) + \psi_{il}^c(b)\sin(\nu_{il}t)\end{aligned}$$

for some $q \in \mathbb{N}$, where for all $1 \leq l \leq q$, $0 \neq \nu_{il} \in \mathbb{Q}$, and $\psi_{il}^a(b)$ and $\psi_{il}^c(b)$ are $\mathcal{R}$-terms.

Thus, $\Phi(x,u,t)$ can be written component-wise as follows, for all $1 \leq i \leq n$

$$\Phi(x,u,t)_i = \sum_{k=1}^{p} \phi_{ik}^a(b,x)\cos(\rho_{ik}t) + \sum_{k=1}^{p} \phi_{ik}^c(b,x)\sin(\rho_{ik}t) = \widehat{\Phi}(x,b,t)_i$$

for some $p \in \mathbb{N}$, where for all $1 \leq k \leq p$, $0 \neq \rho_{ik} \in \mathbb{Q}$, and $\phi_{ik}^a(b,x)$ and $\phi_{ik}^c(b,x)$ are $\mathcal{R}$-terms. Now, for each $1 \leq k \leq p$ let d_{ik} denote the denominator of ρ_{ik} and let $d = \prod_{i=1}^{n}\prod_{k=1}^{p} d_{ik}$. We assume that the ρ_{ik} are in reduced form, with positive denominators. Then $d > 0$ and for each $1 \leq k \leq p$ we write $\delta_{ik} = \rho_{ik}d$. Then by using the change of variable $t = ds$ we have:

$$\widetilde{\Phi}(x,b,s)_i = \widehat{\Phi}(x,b,t)_i[ds/t] = \sum_{k=1}^{p} \phi_{ik}^a(b,x)\cos(\delta_{ik}s) + \sum_{k=1}^{p} \phi_{ik}^c(b,x)\sin(\delta_{ik}s).$$

The following lemma can easily be proved using simple trigonometric identities.

LEMMA 3.6. *For each $m \geq 1$ there exist homogeneous polynomials $f_m(x,y)$ and $g_m(x,y)$ of degree m such that $\cos(ms) = f_m(\cos s, \sin s)$ and $\sin(ms) = g_m(\cos s, \sin s)$.*

Using Lemma 3.6 we have that, for all $1 \le i \le n$,

$$\widetilde{\Phi}(x,b,s)_i = \sum_{k=1}^{p} \phi_{ik}^a(b,x) f_{|\delta_{ik}|}(\cos s, \mathrm{sg}(\delta_{ik}) \sin s) + \phi_{ik}^c(b,x) g_{|\delta_{ik}|}(\cos s, \mathrm{sg}(\delta_{ik}) \sin s).$$

Using the trigonometric equation $\cos^2 s + \sin^2 s = 1$ and the change of variables $z = \cos s$ and $w = \sin s$, we obtain:

$$\widetilde{\Phi}(x,b,s)_i = \overline{\Phi}(x,b,z,w)_i = \sum_{k=1}^{p} \phi_{ik}^a(b,x) f_{|\delta_{ik}|}(z, \mathrm{sg}(\delta_{ik}) w) + \phi_{ik}^c(b,x) g_{|\delta_{ik}|}(z, \mathrm{sg}(\delta_{ik}) w)$$

with $z^2 + w^2 = 1$. This proves the following lemma.

Lemma 3.7. *If $\mathcal{F}$ is a family of vector fields such that*

(1) $A \in \mathbb{Q}^{n\times n}$ is diagonalizable, and has purely imaginary eigenvalues of the form $\mathbf{i}r$ with $r \in \mathbb{Q}$, and
(2) $\mathcal{U}$ is defined by equation (3.5) with $p_l(t) = \sin(\mu_l t)$ or $p_l(t) = \cos(\mu_l t)$, $\mathbf{i}\mu_l \notin \Lambda$, $\mu_l \in \mathbb{Q}$,

then there exists a vector $\overline{\Phi}$ of polynomials $\overline{\Phi}_i \in \mathbb{Q}[x,b,z,w]$ such that for all $t \ge 0$ and $u \in \mathcal{U}$, $\Phi(x,u,t) \equiv \overline{\Phi}(x,b,z,w)$ with $z^2 + w^2 = 1$.

Let $Y = \{y \in \mathbb{R}^n \mid \Phi_Y(y)\}$. Then Lemma (3.7) implies that $\mathrm{Pre}(Y)$ is equivalent to:

$$\mathrm{Pre}(Y) = \Big\{ x \in \mathbb{R}^n \mid \exists y\, \exists b\, \exists z\, \exists w : \Phi_Y(y) \wedge \Gamma(b) \wedge z^2 + w^2 = 1 \wedge \bigwedge_{i}^{n} y_i = \overline{\Phi}(x,b,z,w)_i \Big\}. \tag{3.18}$$

We have therefore proved the following proposition.

Proposition 3.8. *Let $Y \subseteq \mathbb{R}^n$ be an $\mathcal{R}$-definable set and $\mathcal{F} = (A, \mathcal{U})$ be a family of vector fields such that*

(1) $A \in \mathbb{Q}^{n\times n}$ is diagonalizable, and has purely imaginary eigenvalues of the form $\mathbf{i}r$ with $r \in \mathbb{Q}$, and
(2) $\mathcal{U}$ is defined by equation (3.5) with $p_l(t) = \sin(\mu_l t)$ or $p_l(t) = \cos(\mu_l t)$, $\mathbf{i}\mu_l \notin \Lambda$, $\mu_l \in \mathbb{Q}$,

Then, Pre(Y) and Post(Y) are definable in $\mathcal{R}$. Moreover, they are computable.

Example 3.6. Consider the control linear vector field given by the matrix $A \in \mathbb{Q}^{2\times 2}$ and $\mathcal{U} = \{u\}$ defined as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}, \qquad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}.$$

The eigenvalues of A are $\pm\mathbf{i}2$ which result from solving the equation $\det(A - \lambda I) = \lambda^2 + 4 = 0$ for the unknown λ. The matrices D, T and T^{-1} such that $A = TDT^{-1}$ are:

$$D = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

Using the ODEMATSYS solver of MACSYMA (MIT The Mathlab Group, 1983) we obtain:

$$\Phi(x, u, t) = \begin{bmatrix} x_1(\cos^2(t) - \sin^2(t)) + \frac{1}{3}(3x_2 + 5)\cos(t)\sin(t) - \frac{2}{3}\sin(t) \\ \frac{1}{3}(3x_2 + 5)(\cos^2(t) - \sin^2(t)) - 4x_1\cos(t)\sin(t) - \frac{5}{3}\cos(t) \end{bmatrix}.$$

After substituting w by $\sin(t)$ and z by $\cos(t)$ in the above expression we obtain:

$$\overline{\Phi}(x, z, w) = \begin{bmatrix} x_1(z^2 - w^2) + \frac{1}{3}(3x_2 + 5)zw - \frac{2}{3}w \\ \frac{1}{3}(3x_2 + 5)(z^2 - w^2) - 4x_1 zw - \frac{5}{3}z \end{bmatrix}.$$

Let $X = \{(1, -\frac{5}{3})\}$ be the initial set and $Y = \{(y_1, y_2) \mid y_1 = 0 \wedge y_2 > 0\}$ the set to be reached. We have that:

$$Y \cap \mathrm{Post}(X) = \Big\{(0, y_2) \in \mathbb{R}^2 \mid \exists w\, \exists z : y_2 > 0$$
$$\wedge w^2 + z^2 = 1 \wedge 0 = (z^2 - w^2) - \frac{2}{3}w \wedge y_2 = -4zw - \frac{5}{3}z\Big\}.$$

Attempting to compute a quantifier-free expression for $Y \cap \mathrm{Post}(X)$ using either REDLOG or QEPCAD alone does not work. However, it is possible to do it by combining both tools as follows. We first eliminate z with REDLOG obtaining:

$$\begin{aligned} Y \cap \mathrm{Post}(X) = \{(0, y_2) \in \mathbb{R}^2 \mid \exists w : y_2 > 0 & \\ \wedge\, 432w^4 + 648w^3 + 315w^2 + 50w - 27y_2^2 = 0 & \\ \wedge\, 144w^4 + 120w^3 - 119w^2 - 120w + 9y_2^2 - 25 = 0\}, & \end{aligned}$$

with the assumption that $12w + 5 \neq 0$. We can verify with REDLOG that $Y \cap \mathrm{Post}(X) = \emptyset$ if $12w + 5 = 0$. We then use QEPCAD to eliminate w from the result. We obtain:

$$Y \cap \mathrm{Post}(X) = \{(0, y_2) \in \mathbb{R}^2 \mid y_2 > 0 \wedge 2916y_2^4 - 32688y_2^2 + 22445 = 0\}.$$

This set is shown to be non-empty by further eliminating y_2 with QEPCAD. Checking non-emptiness can also be done by finding whether the polynomial $p(y_2) = 2916y_2^4 - 32688y_2^2 + 22445$ has positive real roots. We have done this using the functionalities provided by the computer-algebra package REDUCE (Hearn, 1999) and obtained:

$$y_2 = \frac{\sqrt{-181\sqrt{19} + 908}}{9\sqrt{2}} \approx 0.857211, \qquad y_2 = \frac{\sqrt{181\sqrt{19} + 908}}{9\sqrt{2}} \approx 3.23653.$$

The graph of $p(y_2)$ in the interval $[0, 4]$ is depicted in Figure 5.†

EXAMPLE 3.7. Consider the control linear vector field given by the matrix $A \in \mathbb{Q}^{2\times 2}$ and $\mathcal{U}$ defined as follows:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} a\cos(2t) \\ -a^{-1}\sin(2t) \end{bmatrix}, \qquad \text{with } a > 0.$$

†Following the same approach of combining REDLOG and QEPCAD we indeed obtained a quantifier-free expression of the set $\mathrm{Post}(X)$. The result is a Boolean combination of 18 atomic formulae and is too large to be shown here.

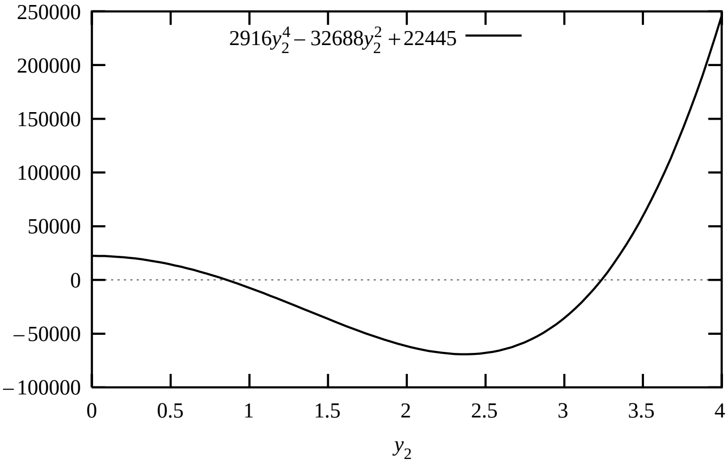

Figure 5. Result of Example 3.6.

The eigenvalues of A are $\pm\mathbf{i}$. Using the ODEMATSYS solver of MACSYMA we obtain:

$$\Phi(x,u,t) = \frac{1}{3a}\begin{bmatrix} (4a^2-2)\sin(2t)\cos(2t) + 3ax_1\cos(2t) - (a^2-2+3ax_2)\sin(2t) \\ (2-a^2)(\cos(2t)^2 - \sin(2t)^2) + (a^2-2+3ax_2)\cos(2t) + 3ax_1\sin(2t) \end{bmatrix}.$$

After substituting w by $\sin(2t)$ and z by $\cos(2t)$ in the above expression we obtain:

$$\overline{\Phi}(x,z,w) = \frac{1}{3a}\begin{bmatrix} (4a^2-2)wz + 3ax_1z - (a^2-2+3ax_2)w \\ (2-a^2)(z^2-w^2) + (a^2-2+3ax_2)z + 3ax_1w \end{bmatrix}.$$

Let the initial set be $X = \{(0,0)\}$ and the final set be $Y = \{(-1,1)\}$. We have that $Y \cap \mathrm{Post}(X) \neq \emptyset$ iff the following formula holds:

$$\exists w\ \exists z\ \exists a:\ a > 0 \wedge w^2 + z^2 = 1$$
$$\wedge -1 = \frac{w}{3a}\left((4a^2-2)z + 2 - a^2\right) \wedge 1 = \frac{a^2-2}{3a}\left(w^2 - z^2 + z\right).$$

Neither REDLOG nor QEPCAD alone are able to compute an equivalent quantifier-free expression. However, it is possible to do it by combining both tools as in Example 3.6. We first eliminate w with REDLOG obtaining:

$$\begin{aligned} &\exists z\ \exists a:\ a > 0 \\ &\quad \wedge 16a^6z^4 - 24a^6z^3 + 9a^6z^2 - a^6z + 48a^5z^2 - 24a^5z + 3a^5 - 48a^4z^4 + 84a^4z^3 \\ &\qquad -42a^4z^2 + 6a^4z - 9a^4 - 48a^3z^2 + 60a^3z - 12a^3 + 36a^2z^4 - 84a^2z^3 + 60a^2z^2 \\ &\qquad -12a^2z + 18a^2 + 12az^2 - 24az + 12a - 8z^4 + 24z^3 - 24z^2 + 8z = 0 \\ &\quad \wedge 16a^4z^4 - 8a^4z^3 - 15a^4z^2 + 8a^4z - a^4 - 16a^2z^4 + 20a^2z^3 + 12a^2z^2 - 20a^2z \\ &\qquad +13a^2 + 4z^4 - 8z^3 + 8z - 4 = 0 \end{aligned}$$

with the assumption that $4a^2z - a^2 - 2z + 2 \neq 0$. This formula is too complex to be able to automatically eliminate the quantifiers. However, if we set z to be 0, the above formula holds provided that:

$$\exists a:\ a > 0 \wedge (a^2 \neq 2) \wedge 3a^4 - 9a^3 - 12a^2 + 18a + 12 = 0 \wedge -a^4 + 13a^2 - 4 = 0$$

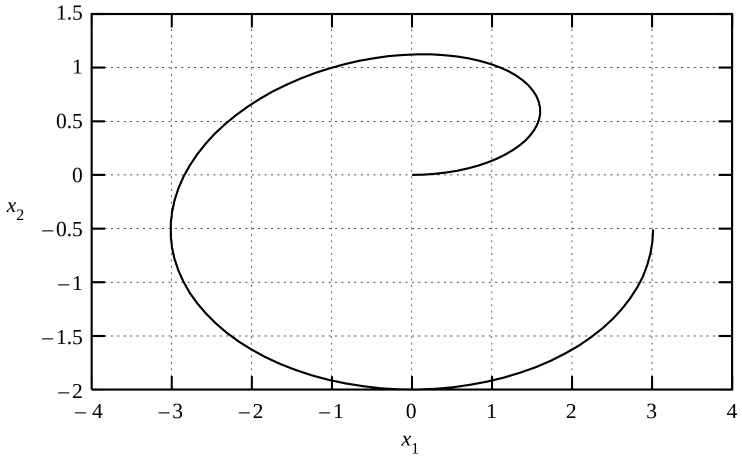

Figure 6. Computed trajectory for Example 3.7.

which is found to be true using QEPCAD. Indeed, we used REDUCE to find the roots of the polynomials in a and found the common root $r = \frac{1}{2}(\sqrt{17} + 3) \approx 3.56156$ satisfying $r > 0$ and $r^2 > 2$. Hence, $y = (-1, 1)$ is reachable from $x = (0, 0)$ by taking $a = r$. Figure 6 depicts the trajectory.

3.4. MAIN DECIDABILITY RESULT

Propositions 3.3, 3.5, and 3.8, collectively prove the following theorem.

THEOREM 3.9. (REACHABILITY COMPUTATIONS FOR FAMILIES OF LINEAR SYSTEMS) *Let $\mathcal{F} = (A, \mathcal{U})$ be a family of linear control vector fields with $A \in \mathbb{Q}^{n \times n}$, $\mathcal{U}$ defined by equation* (3.5)*, and one of the following cases holds:*

(1) A is nilpotent, and the basis functions are $p_l(t) = t^l$, or
(2) A is diagonalizable with real, rational eigenvalues, and the basis functions are $p_l(t) = e^{\mu_l t}$, with $\mu_l \notin \Lambda$, $\mu_l \in \mathbb{Q}$, or
(3) A is diagonalizable, has purely imaginary eigenvalues of the form $\mathbf{i}r$ with $r \in \mathbb{Q}$, and the basis functions are $p_l(t) = \sin(\mu_l t)$ or $p_l(t) = \cos(\mu_l t)$, with $\mathbf{i}\mu_l \notin \Lambda$, $\mu_l \in \mathbb{Q}$.

Then, the reachability Problem 3.1 *for the family $\mathcal{F} = (A, \mathcal{U})$ is decidable.*

In the case where there are no control inputs, but simply linear vector fields of the form $\dot{\xi} = A\xi$, then no resonance conditions need to be imposed. This allows us to obtain, as a corollary, the following result which was proven in Lafferriere *et al.* (1999a).

COROLLARY 3.10. (REACHABILITY COMPUTATIONS FOR LINEAR SYSTEMS) *Let $A \in \mathbb{Q}^{n \times n}$, and one of the following cases holds:*

(1) A is nilpotent, or

(2) A is diagonalizable with real, rational eigenvalues, or
(3) A is diagonalizable, and has purely imaginary eigenvalues of the form $\mathbf{i}r$ *with* $r \in \mathbb{Q}$.

Then, the reachability problem for the vector field $\dot{\xi} = A\xi$ *is decidable.*

A slight extension of Theorem 3.9 is possible by allowing the system matrix A to belong to a set of matrices $\mathcal{A}$, *as long as the eigenstructure of A remains the same*. For example, we can allow the rational entries of A to satisfy certain semi-algebraic constraints as long as the matrix remains nilpotent. Also the assumptions of Theorem 3.9 can be relaxed a little by allowing real eigenvalues (as opposed to rational) *as long as the eigenvalues are rationally related*. For example, $\lambda = \sqrt{2}$ can be allowed as long as all the other eigenvalues are rational multiples of λ.

4. Conclusions

In this paper, we presented the first known *families* of linear vector fields whose reachable spaces can be computed exactly. Indeed, we have identified fragments of the real field extended with exponential and trigonometric functions that admit quantifier elimination by applying an appropriate change of variables. This approach allows us to perform computations using quantifier elimination techniques in the decidable theory of the reals. Other decidable fragments of the real field extended with transcendental functions have been found in Anai and Weispfenning (2000) and Weispfenning (2000). They provide procedures for deciding the truth value of formulae of the form $\exists x.\phi(x)$, where $\phi(x)$ is a quantifier-free formula. Although these fragments allow more complex expressions involving transcendental functions, they do not admit quantifier elimination, which is a fundamental property of our framework.

Our result has allowed us to use tools such as Redlog and Qepcad, together with computer-algebra systems (e.g. Macsyma, Reduce), in order to demonstrate various reachability computations for three distinct families of linear vector fields. Such computations can be incorporated in state of the art model checking and deductive verification tools for hybrid systems.

It would be of great interest to use quantifier elimination in order to perform reachability computations of linear vector fields with arbitrary eigenstructure. Unfortunately, the approach of Section 3 does not apply to arbitrary eigenvalues. For such cases, it would be useful to *over-approximate* the reachable sets of linear systems with arbitrary eigenvalues by reachable sets of the decidable families. This idea can also be applied to some classes of non-linear systems.

Acknowledgements

We thank Gerardo Schneider for his help in using Reduce and Redlog. This research has been supported by the DARPA MoBIES grant F33615-00-C-1707, by the European Project ESPRIT-LTR 26270 VHS, and by Projet IMAG MASH.

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Received 16 May 2000
Accepted 9 April 2001